Hyperbolic Structures in Models for Compressible Two-Phase Flow

Macroscopic Modeling of Vehicular and Pedestrian Traffic, Reggio Emilia, February 14-15, 2019

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Content of the Lecture

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5. Exploiting the Hyperbolic Structure of the Relaxed DI Model
1. The Diffuse Interface Concept
1. The Diffuse Interface Concept
For the mathematical modelling different spatial scales can be chosen.

Depending on the scale there are different definitions of phases and interfaces.
1. The Diffuse Interface Concept

From Sharp Interface to Diffuse Interface models for resolved flow:

Sharp Interface

Diffuse Interface
1. The Diffuse Interface Concept

Possible advantages of the DI concept:

- Topological changes (e.g. merging of droplets) can be covered.
- Nucleation can be (easily) modelled in the framework of the DI concept.
- Mixture states that are excluded for SI concepts might have some physical relevance.
- The numerical analysis avoids a complex interface tracking.
- The rigorous passage from resolved to homogenized flow scenarios can be analyzed.
2. Thermodynamical Setting and Equilibria
2. Thermodynamical Setting and Equilibria

We consider a simplified isothermal situation at fixed temperature $T^* > 0$. In this case

- the pressure $p = p(\rho)$,
- the Helmholtz free energy $W = W(\rho)$ and
- the chemical potential $\mu = \mu(\rho)$

are functions of density $\rho > 0$ alone and related by

$$p'(\rho) = \rho W''(\rho), \quad \mu(\rho) = W'(\rho) \quad (\Rightarrow p(\rho) = -W(\rho) + \rho\mu(\rho)).$$
2. Thermodynamical Setting and Equilibria

The Van-der-Waals pressure function is given by

\[ p(\rho) = \frac{\rho R T^*_c}{b - \rho} - a \rho^2. \]

For the subcritical case \( T^*_c < T_{\text{crit}} \) we obtain

a non-monotone graph for \( p \) and a double-well structure for \( W \).

(a) \( T^*_c < T_{\text{crit}} : \text{sgn}(p') \in \{-1, 0, 1\} \).

(b) \( T^*_c < T_{\text{crit}} : \text{sgn}(W'') \in \{-1, 0, 1\} \).
2. Thermodynamical Setting and Equilibria

For $T_* < T_{\text{crit}}$ the fluid occurs in a liquid and a vapour state.

**Definition (phases)**

The fluid is in the vapour (liquid) state iff we have

$$\rho \in \mathcal{A}_{\text{vap}} := (0, \alpha_1) \quad (\rho \in \mathcal{A}_{\text{liq}} := (\alpha_2, b)).$$

**Definition (Maxwell states)**

The states $\beta_2, \beta_2 \in (0, b)$ with

$$W'(\beta_1) = W'(\beta_2) = \frac{W(\beta_2) - W(\beta_1)}{\beta_2 - \beta_1}$$

are called Maxwell states.
3. The Local Diffuse Interface Model (Navier-Stokes-Korteweg Equations)
3. The local DI model

Local diffuse interface equilibria (Van-der-Waals functional):

For $\varepsilon > 0$ find a minimizer $\rho^\varepsilon \in \mathcal{A}(m) \cap H^1(D)$ of

$$F_{\text{local}}^{\varepsilon}[\rho^\varepsilon] := \int_D W(\rho^\varepsilon(x)) + \frac{\varepsilon^2}{2} |\nabla \rho^\varepsilon(x)|^2 \, dx.$$ 

**Theorem: (Modica ’87)**

Let $\{\rho^\varepsilon\}_{\varepsilon > 0} \subset \mathcal{A}(m) \cap H^1(D)$ be a sequence of minimizers of $F_{\text{local}}^{\varepsilon}$.

For a subsequence there exists a function $\rho \in BV(D, \{\beta_1, \beta_2\}) \cap \mathcal{A}(m)$ with

$$\lim_{\varepsilon \to 0} \|\rho - \rho^\varepsilon\|_{1,D} = 0.$$ 

Moreover, $\rho$ is a static equilibrium with the Maxwell states $(\beta_1, \beta_2)$.

**Euler-Lagrange equation:**

$$-\varepsilon^2 \Delta \rho^\varepsilon + W'(\rho^\varepsilon) = C_\varepsilon \text{ in } D.$$
3. The local DI model

Numerical experiment\(^1\): (Static equilibria)

\[
d = 2, \quad D = B_1(0),
\]

\[
\varepsilon = \sqrt{0.001}
\]

\[
d = 3, \quad D = B_1(0) \times (-0.1, 0.1),
\]

\[
\varepsilon = \sqrt{0.001}
\]

---

3. The local DI model

The local Navier-Stokes-Korteweg model (Dunn&Serrin ’85):
(Non-conservative (capillarity) formulation)

\[ \rho_t^\varepsilon + \text{div}(\rho^\varepsilon \mathbf{v}^\varepsilon) = 0 \]
\[ (\rho^\varepsilon \mathbf{v}^\varepsilon)_t + \text{div}(\rho^\varepsilon \mathbf{v}^\varepsilon \otimes \mathbf{v}^\varepsilon + p(\rho^\varepsilon) \mathcal{I}) = \text{div}(\mathcal{T}^\varepsilon) + \rho^\varepsilon \nabla D^\varepsilon[\rho^\varepsilon] \text{ in } \mathbb{R}_{>0} \times D, \]
\[ D^\varepsilon[\rho^\varepsilon] = \varepsilon^2 \Delta \rho^\varepsilon \]
\[ \mathbf{v}^\varepsilon = 0, \quad \nabla \rho^\varepsilon \cdot \mathbf{n} = 0 \text{ in } \mathbb{R}_{>0} \times \partial D \]

Viscous part of the stress tensor:

\[ \mathcal{T}^\varepsilon_{ij} := \mu(\varepsilon)\text{div}(\mathbf{v}^\varepsilon)\delta_{ij} + 2\nu(\varepsilon)\mathcal{D}_{ij}, \quad \mathcal{D}_{ij} := \frac{1}{2}(\mathbf{v}_{j,x_i}^\varepsilon + \mathbf{v}_{i,x_j}^\varepsilon). \]
3. The local DI model

Entropy consistency:

Any classical solution of the initial boundary value problem for the local DI model satisfies

\[
\frac{d}{dt}\left(F_{\text{local}}^{\varepsilon}[\rho^{\varepsilon}(t, \cdot)] + \int_{\Omega} 2\rho^{\varepsilon}(t, \cdot)|\bm{v}^{\varepsilon}(t, \bm{x})|d\bm{x}\right) \\
\leq -\int_{\Omega} 2\nu\mathcal{D}(\bm{v}^{\varepsilon}(t, \bm{x})) : \mathcal{D}(\bm{v}(t, \bm{x})) + \mu(\text{div}(\bm{v}^{\varepsilon}(t, \bm{x})))^2 d\bm{x} \leq 0.
\]

Note:

For the existence of weak and classical solutions see e.g. Bresch et al. ’03 and Hattori ’96.
3. The local DI model

Numerical experiment$^2$: (Evolution of randomly initialized bubble ensemble)

\[ \varepsilon = \sqrt{0.0001}, \mu(\varepsilon) = \varepsilon, \nu(\varepsilon) = -\frac{2}{3}\mu. \]

Density distribution at times $t = 0.0, 0.2, 1.0, 4.0, 15.0, 100.0$.

---

3. The local DI model

Sharp interface limit $\varepsilon \to 0 (d = 1)$:

Initial configuration:

$$
\begin{align*}
\rho_0(x) &= \begin{cases} 
1.8, & x \in (0.3, 0.6) \\
0.3, & \text{else} 
\end{cases} \\
v_0(x) &= 0.
\end{align*}
$$

Density for the local DI model at $t = 1.72$ with mesh size $h = 0.005$:

- $\varepsilon = 0.01 > h$
- $\varepsilon = 0.001 < h$
- $\varepsilon = 0.0001 \ll h$

Computation crashed.
3. The local DI model

Sharp interface limit $\varepsilon \to 0 \ (d = 1)$:

Initial configuration:

\[
\begin{align*}
\rho_0(x) & = \begin{cases} 
1.8, & x \in (0.3, 0.6) \\
0.3, & \text{else}
\end{cases} \\
\nu_0(x) & = 0.
\end{align*}
\]

Evolution of generalized entropy for the local DI model with $\varepsilon = 0.01$: 

![Graph showing evolution of generalized entropy with different resolutions.](image)
3. The local DI model

Hyperbolic-elliptic structure of the first-order Euler part:

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{v}) &= 0, \\
(\rho \mathbf{v})_t + \text{div}(\rho \mathbf{v} \otimes \mathbf{v} + p(\rho) \mathbf{I}) &= \text{div}(\mathcal{T}^\varepsilon) + \rho \nabla D[\rho], \\
D[\rho] &= \varepsilon^2 \Delta \rho.
\end{align*}
\]

Characteristic structure of the first-order part ($d=2$):
The eigenvalues of the flux Jacobian are given for some $\xi \in S^1$ by

\[
\begin{align*}
\lambda_1(\rho, \mathbf{v}) &= \mathbf{v} \cdot \xi - \sqrt{p'(\rho)}, \\
\lambda_2(\rho, \mathbf{v}) &= \mathbf{v} \cdot \xi, \\
\lambda_3(\rho, \mathbf{v}) &= \mathbf{v} \cdot \xi + \sqrt{p'(\rho)}.
\end{align*}
\]

\[\sim\] The discretization of the first order flux is basically restricted to central fluxes.
\[\sim\] The non-monotone pressure prevents $L^p$-type a-priori estimates.
4. Relaxed and Nonlocal Diffuse Interface Models
4. Relaxed and Nonlocal Diffuse Interface Models

Relaxed diffuse interface equilibria (Brandon&Lin&Rogers ’95):

For $\varepsilon, \alpha > 0$ find a minimizer $(\rho^{\varepsilon,\alpha}, c^{\varepsilon,\alpha}) \in (\mathcal{A}(m) \cap L^2(D)) \times H^1(D))$ of

$$F^{\varepsilon,\alpha}_{\text{order}}[\rho^{\varepsilon,\alpha}, c^{\varepsilon,\alpha}] := \int_D \left( W(\rho^{\varepsilon,\alpha}(x)) + \frac{\alpha}{2}(\rho^{\varepsilon,\alpha} - c^{\varepsilon,\alpha})^2 + \frac{\varepsilon^2}{2} |\nabla c^{\varepsilon,\alpha}|^2 \right) dx.$$ 

Euler-Lagrange system:

$$-\alpha(c^{\varepsilon,\alpha} - \rho^{\varepsilon,\alpha}) + W'(\rho^{\varepsilon,\alpha}) = C^{\varepsilon,\alpha},$$

$$\frac{\varepsilon^2}{\alpha} \Delta c^{\varepsilon,\alpha} = c^{\varepsilon,\alpha} - \rho^{\varepsilon,\alpha}$$

in $D$

Note: A Modica-type result for the SI limit $\varepsilon \to 0$ with $\alpha$ fixed is given in Solci&Vitali ’03.
4. Relaxed and Nonlocal Diffuse Interface Models

The Korteweg limit for the relaxed DI model:

Let $\varepsilon > 0$ be fixed and let

$$\{(\rho^\alpha, c^\alpha)\}_{\alpha > 0} := \{(\rho^{\varepsilon, \alpha}, c^{\varepsilon, \alpha})\}_{\alpha > 0}$$

be a sequence of minimizers of $F_{\text{relax}}^{\varepsilon, \alpha}$.

**Theorem:**

If $\{(\rho^\alpha, c^\alpha)\}_{\alpha > 0}$ converges for $\alpha \to \infty$ a.e. to a function

$$(\rho, c) \in (H^1(D))^2,$$

the limit function $\rho$ is a minimizer of $F_{\text{local}}^\varepsilon$ and we have $c = \rho$.

**Proof:** cf. Solci ’03 and Brandon&Lin&Rogers ’95.

**Note:** The relaxed DI model is an approximate model for the local DI model.
4. Relaxed and Nonlocal Diffuse Interface Models

General nonlocal Van-der-Waals model:
Consider for a kernel function $K : \mathbb{R}^d \to [0, \infty)$ the scaled version

$$K^{\varepsilon, \alpha}(x) = \left( \frac{\alpha}{\sqrt{\varepsilon}} \right)^d K \left( \frac{\sqrt{\alpha}}{\varepsilon} x \right).$$

For $\varepsilon, \alpha > 0$ find a minimizer $\rho^{\varepsilon, \alpha} \in (A(m) \cap L^2(D)))$ of

$$F_{\text{nonlocal}}^{\varepsilon, \alpha}[\rho^{\varepsilon, \alpha}] := \int_D \left( W(\rho^{\varepsilon, \alpha}(x)) + \alpha \int_D K^{\varepsilon, \alpha}(x - y)(\rho^{\varepsilon, \alpha}(x) - \rho^{\varepsilon, \alpha}(y))^2 \, dy \right) \, dx.$$

Euler-Lagrange equation:

$$-\left( K^{\varepsilon, \alpha} \ast \rho^{\varepsilon, \alpha} - \rho^{\varepsilon, \alpha} \right) + W'(\rho^{\varepsilon, \alpha}) = C_{\varepsilon, \alpha} \text{ in } D$$

Note: A Modica-type theorem has been proven in Alberti & Bellettini '98. The Korteweg limit is considered (indirectly) in Alberti et al. '96.
4. Relaxed and Nonlocal Diffuse Interface Models

Relaxed DI model as nonlocal model for $D = \mathbb{R}^3$:

Euler-Lagrange system for the relaxed DI model:

$$
-\alpha (c^{\varepsilon,\alpha} - \rho^{\varepsilon,\alpha}) + W'(\rho^{\varepsilon,\alpha}) = C_{\varepsilon,\alpha}
$$

$$
\frac{\varepsilon^2}{\alpha} \Delta c^{\varepsilon,\alpha} = c^{\varepsilon,\alpha} - \rho^{\varepsilon,\alpha}
$$

in $D$

The screened Poisson equation for $c^{\varepsilon,\alpha}$ can be explicitly solved by

$$
c^{\varepsilon,\alpha} = K^{\varepsilon,\alpha} * \rho^{\varepsilon,\alpha},
$$

using $\nu = \varepsilon / \sqrt{\alpha}$ and

$$
K^{\varepsilon,\alpha}(x) = \frac{1}{\nu^3} K\left(\frac{x}{\nu}\right),
$$

$$
K(x) = \frac{1}{4\pi} \frac{e^{-|x|}}{|x|}.
$$
4. Relaxed and Nonlocal Diffuse Interface Models

Relaxed DI model as nonlocal model for $D = \mathbb{R}^3$:

The Euler-Lagrange system for the relaxed DI model reduces then to an integral equation for $\rho^{\varepsilon,\alpha}$, i.e.

$$-\alpha(K^{\varepsilon,\alpha} * \rho^{\varepsilon,\alpha} - \rho^{\varepsilon,\alpha}) + W'(\rho^{\varepsilon,\alpha}) = C_{\varepsilon,\alpha} \text{ in } D.$$ 

The relaxed DI model is a nonlocal DI model with special weakly integrable kernel.

**Note:** A bounded domain $D$ does not allow the explicit computation of the solution as a convolution.
4. Relaxed and Nonlocal Diffuse Interface Models

The relaxed Navier-Stokes-Korteweg model:$$^3$$

\[
\begin{aligned}
\rho_t^{\varepsilon,\alpha} + \text{div}(\rho^{\varepsilon,\alpha}v^{\varepsilon,\alpha}) &= 0 \\
(\rho^{\varepsilon,\alpha}v^{\varepsilon,\alpha})_t + \text{div}(\rho^{\varepsilon,\alpha}v^{\varepsilon,\alpha} \otimes v^{\varepsilon,\alpha} + p(\rho^{\varepsilon,\alpha})I) &= \text{div}(T^{\varepsilon}) \\
-\varepsilon^2 \Delta c^{\varepsilon,\alpha} &= \alpha(\rho^{\varepsilon,\alpha} - c^{\varepsilon,\alpha}) \\
v^{\varepsilon,\alpha} &= 0, \quad \nabla c^{\varepsilon,\alpha} \cdot n = 0
\end{aligned}
\]

in \(\mathbb{R}_{>0} \times D\),

\[
\frac{d}{dt} \left( F_{\text{Relax}}^{\varepsilon,\alpha}[\rho^{\varepsilon,\alpha}(t, \cdot), c^{\varepsilon,\alpha}(t, \cdot)] + \int_D \frac{1}{2} \rho^{\varepsilon,\alpha}(t, x)|v^{\varepsilon,\alpha}(t, x)|^2 \, dx \right) \leq 0
\]

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4. Relaxed and Nonlocal Diffuse Interface Models

Nonlocal DI model\(^4\):

\[
\begin{align*}
\rho_t^\varepsilon,\alpha + \text{div}(\rho^\varepsilon,\alpha \mathbf{v}^\varepsilon,\alpha) &= 0 \\
(\rho^\varepsilon,\alpha \mathbf{v}^\varepsilon,\alpha)_t + \text{div}(\rho^\varepsilon,\alpha \mathbf{v}^\varepsilon,\alpha \otimes \mathbf{v}^\varepsilon,\alpha + p(\rho^\varepsilon,\alpha)\mathbf{I}) &= \text{div}(\mathbf{T}^\varepsilon) \\
&\quad + \alpha \rho^\varepsilon,\alpha \nabla(\mathbf{K}^\varepsilon,\alpha \ast \rho^\varepsilon,\alpha - \rho^\varepsilon,\alpha) \\
\mathbf{v}^\varepsilon,\alpha &= 0
\end{align*}
\]

in \(\mathbb{R}_{>0} \times D\),

\[
dt \left( F_{\text{nonlocal}}^\varepsilon,\alpha[\rho^\varepsilon,\alpha(t, \cdot), c^\varepsilon,\alpha(t, \cdot)] + \int_D \frac{1}{2} \rho^\varepsilon,\alpha(t, \mathbf{x}) |\mathbf{v}^\varepsilon,\alpha(t, \mathbf{x})|^2 \, d\mathbf{x} \right) \leq 0
\]

4. Relaxed and Nonlocal Diffuse Interface Models

Qualitative Test/Bubble Merging\(^5\):

4. Relaxed and Nonlocal Diffuse Interface Models

Recall the 1D numerical experiment of Section 3 with the local DI model:

\[
\begin{align*}
\rho_0(x) &= \begin{cases} 
1.8, & x \in (0.3, 0.6) \\
0.3, & \text{else}
\end{cases} \\
v_0(x) &= 0.
\end{align*}
\]

Density for the local DI model at \( t = 1.72 \) with mesh size \( h = 0.005 \):

\[
\varepsilon = 0.01 > h \
\varepsilon = 0.001 < h \
\varepsilon = 0.0001 \ll h
\]

computation crashed
4. Relaxed and Nonlocal Diffuse Interface Models

Density for the relaxed DI model at $t = 1.72$ (with $\alpha = 100$):

- $\varepsilon = 0.01$
- $\varepsilon = 0.001$
- $\varepsilon = 0.0001$

Energy evolution for the relaxation approximation with $\varepsilon = 0.01$: 
4. Relaxed and Nonlocal Diffuse Interface Models

High density ratio:

\[ h = 0.0025, \varepsilon = 0.01 \]

Density evolution for the relaxation approximation for \( \alpha = 100, 1000 \):
4. Relaxed and Nonlocal Diffuse Interface Models

High density ratio:

\[ h = 0.0025, \varepsilon = 0.01 \]

Density evolution for the relaxation approximation for \( \alpha = 100, 1000 \):
4. Relaxed and Nonlocal Diffuse Interface Models

Relaxation approximation and hyperbolicity: (indices $\varepsilon, \alpha$ skipped, $d = 1$)

\[
\begin{align*}
\rho_t + (\rho v)_x &= 0 \\
(\rho v)_t + (\rho v^2 + p(\rho))_x &= \varepsilon v_{xx} + \alpha \rho (c - \rho)_x \\
-\varepsilon^2 c_{xx} &= \alpha (\rho - c)
\end{align*}
\]
4. Relaxed and Nonlocal Diffuse Interface Models

Relaxation approximation and hyperbolicity (indices \( \varepsilon, \alpha \) skipped, \( d = 1 \)):

\[
\rho_t + (\rho v)_x = 0
\]

\[
(\rho v)_t + \left( \rho v^2 + \left( p(\rho) + \frac{\alpha}{2} \rho^2 \right) \right)_x = \varepsilon v_{xx} + \alpha \rho c_x
\]

\[
-\varepsilon^2 c_{xx} = \alpha (\rho - c)
\]
4. Relaxed and Nonlocal Diffuse Interface Models

Relaxation approximation and hyperbolicity (indices \( \varepsilon, \alpha \) skipped, \( d = 1 \)):

\[
\begin{pmatrix}
\rho \\
\rho v
\end{pmatrix}
t + f_\alpha \left( \begin{pmatrix}
\rho \\
\rho v
\end{pmatrix}
\right)_x = \begin{pmatrix}
0 \\
\varepsilon v_{xx} + \alpha \rho c_x
\end{pmatrix}
\]

\[ -\varepsilon^2 c_{xx} = \alpha (\rho - c) \]

For a Van-der-Waals pressure the new first-order part is strictly hyperbolic for \( \alpha > \max\{ -W''(\rho) \} \). The eigenvalues of \( Df_\alpha \) are given by

\[
\lambda_1(\rho, v) = v - \sqrt{p'_\alpha(\rho)} \quad \text{and} \quad \lambda_2(\rho, v) = v + \sqrt{p'_\alpha(\rho)}.
\]
5. Exploiting the Hyperbolic Structure of the Relaxed DI Model
5. Hyperbolic Structure

A conservative and entropy-stable discretization:
The first-order part of the relaxed DI model for \( d = 1 \) as conservation law writes as

\[
\begin{pmatrix}
\rho \\
m := \rho v
\end{pmatrix}_t + f_\alpha \left( \begin{pmatrix}
\rho \\
m
\end{pmatrix} \right)_x = 0 \iff u_t + f_\alpha(u)_x = 0.
\]

It is equipped with the entropy/entropy flux pair

\[
\eta_\alpha(\rho, m) = W_\alpha(\rho) + \frac{m^2}{2\rho}, \quad q_\alpha(\rho, m) = \frac{m}{\rho} \left( \eta_\alpha(\rho, m) + p_\alpha(\rho) \right).
\]

The convexity of \( \eta_\alpha \) for \( \alpha \gg 1 \) implies that the mapping

\[
\mathbf{u} \mapsto \mathbf{w}(\mathbf{u}) := \nabla \eta_\alpha(\mathbf{u})
\]

is one-to-one. The first-order part can then be rewritten in the form

\[
\mathbf{u}(\mathbf{w})_t + g_\alpha(\mathbf{w})_x = 0.
\]
5. Hyperbolic Structure

**Theorem** (Tadmor ’84)
There is a 2-parameter family of numerical fluxes $g^* = g^*(w, z)$ such that the scheme

$$u'_j(t) = -\frac{1}{\Delta x} \left( g^*_{j+\frac{1}{2}}(t) - g^*_{j-\frac{1}{2}}(t) \right), \quad g^*_{j+\frac{1}{2}}(t) = g^*(w_j(t), w_{j+1}(t)),$$

is entropy-conservative, i.e. there is a numerical entropy flux $q^* = q^*(w, z)$ that satisfies $q^*(w, w) = q_\alpha(w)$ with

$$\eta_\alpha(u_j(t))' = -\frac{1}{\Delta x} \left( q^*_{j+\frac{1}{2}}(t) - q^*_{j-\frac{1}{2}}(t) \right)$$

for all $t \in (0, T)$ and $j \in \mathbb{Z}$. 
5. Hyperbolic Structure

Theorem [Neusser&R. 16]:
There exists at least one Tadmor flux $g^* = g^*(w, z)$ such that for

$$r^*(w, z) = \begin{cases} \frac{g_1^*(w, z)}{w_2} & : \; w_2 \neq 0, \\ \rho(w) & : \; w_2 = 0. \end{cases}$$

the solution of the semi-discrete Finite Volume scheme

$$u'(t) + \frac{1}{h} \left( g_{j+\frac{1}{2}}^* - g_{j-\frac{1}{2}}^* \right) = \frac{\alpha}{h} \left( 0 \right),$$

$$\varepsilon^2 \left( c_{j+1} - 2c_j + c_{j-1} \right) = \alpha(c_j - \rho_j)$$

satisfies the energy equality

$$\frac{d}{dt} \sum_{j \in \mathbb{Z}} \left( \frac{m_j^2}{2\rho_j} + W(\rho_j) + \frac{\alpha}{2} (\rho_j - c_j)^2 + \frac{\varepsilon^2}{2h^2} (c_{j+1} - c_j)^2 \right) = 0.$$
5. Hyperbolic Structure

A numerical example: Let $d = 1$, $\varepsilon = 0.01$, and $h = 0.00125$

$$
\rho_0(x) = \begin{cases} 
1.8 : & x \in (0.3, 0.6) \cup (0.85, 1.05) \\
0.3 : & \text{else}
\end{cases}, \quad v_0(x) = 0
$$

Evolution for relaxed DI model with discrete solution $u_h^{\alpha}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
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<tr>
<td>$|u_h^{\alpha} - u_h|_{L^2}$</td>
<td>1.039e-1</td>
<td>3.333e-2</td>
<td>1.802e-2</td>
<td>1.909e-3</td>
<td>1.683e-4</td>
</tr>
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<td>-</td>
<td>0.708</td>
<td>0.885</td>
<td>0.975</td>
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</tr>
</tbody>
</table>
5. Hyperbolic Structure
Scalar conservation law with screened Poisson equation:
Let \( f \in C^1(\mathbb{R}) \) with \( |f'(u)| \leq L \) and consider

\[
\begin{align*}
    u_t^\alpha + f(u^\alpha)_x &= \varepsilon u_{xx}^\alpha - \alpha(u^\alpha - c^\alpha)_x & \text{in } (0, T) \times \mathbb{R}, \\
    -\varepsilon^2 c_{xx}^\alpha &= \alpha(u^\alpha - c^\alpha) & \text{in } (0, T) \times \mathbb{R}, \quad (P_\alpha) \\
    u^\alpha(\cdot, 0) &= u_0 & \text{in } \mathbb{R}.
\end{align*}
\]

Uniform a-priori estimates for \( t \in (0, T) \):

(i) \( \| \partial_x^k c^\alpha(t, \cdot) \|_{L^2(\mathbb{R})} \leq \| \partial_x^k u^\alpha(t, \cdot) \|_{L^2(\mathbb{R})} \),

(ii) \( \| u^\alpha(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\varepsilon \| u_x^\alpha \|_{L^2((0,t) \times \mathbb{R})}^2 = O(1) \),

(iii) \( \frac{d}{dt} \int_{\mathbb{R}} (F(u^\alpha(t, \cdot)) + \frac{\alpha}{2}(u^\alpha(t, \cdot) - c^\alpha(t, \cdot))^2 + \frac{\varepsilon}{2}(c_x^\alpha(t, \cdot))^2) \, dx \leq 0, \ F' = f. \)
5. Hyperbolic Structure

Korteweg limit $\alpha \to \infty$:

**Theorem:** Let $u_0 \in H^3(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})$ be given. Then we have

(i) For each $\varepsilon, \alpha > 0$ there is a unique classical solution of $(P_\alpha)$.

(ii) There exists a subsequence of $\{(u_\alpha, c_\alpha)\}_{\alpha > 0}$ and a function $u \in L^2((0, T) \times \mathbb{R})$ such that

$$u_\alpha \to u, \quad c_\alpha \to u \quad \text{in } L^2_{\text{loc}}((0, T) \times \mathbb{R}) \text{ for } \alpha \to \infty.$$ 

(iii) The function $u$ is a weak solution of

$$u_t + f(u)_x = \varepsilon u_{xx} + \varepsilon^2 u_{xxx} \quad \text{in } (0, T) \times \mathbb{R},$$

$$u(0, \cdot) = u_0 \quad \text{in } \mathbb{R}.$$ 

**Note:** Korteweg limit for relaxed DI systems in Charve ’13 and Giesselmann ’14.

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5. Hyperbolic Structure

Outline of proof:

a. Use the ($\alpha$-independent) a-priori estimate (ii) to show

$$\|u^\alpha\|_{L^2(0,T;H^3)} = O(1).$$

b. Straightforward calculations lead to

$$\|u_t^\alpha\|_{L^2((0,T) \times \mathbb{R})} \leq C(\|u^\alpha\|_{L^2(0,T;H^2)}) + \alpha \|u_x^\alpha - c_x^\alpha\|_{L^2((0,T) \times \mathbb{R})}$$

$$= O(1) + \varepsilon^2 \|c_{xxx}^\alpha\|_{L^2((0,T) \times \mathbb{R})}$$

A-priori estimate (i) and Step a. imply

$$\|u_t^\alpha\|_{L^2((0,t) \times \mathbb{R})} = O(1).$$

c. Apply Lions-Aubin lemma to deduce $u^\alpha \rightarrow u$ in $L^2_{loc}((0,T) \times \mathbb{R})$.

d. Apply a-priori estimate (iii) to deduce $u^\alpha - c^\alpha \rightarrow 0$ a.e.
5. Hyperbolic Structure
Contour surface of binary collision for smaller We number (with T. Hitz, ’19):

$t = 0.00$

$t = 0.10$

$t = 0.20$

$t = 0.30$

$t = 0.40$

$t = 0.50$

$t = 0.60$

$t = 0.70$

$t = 0.80$
5. Hyperbolic Structure
Contour surface of binary collision for higher Weber number:

$t = 0.00$  $t = 0.10$  $t = 0.20$
$t = 0.30$  $t = 0.40$  $t = 0.50$
$t = 0.60$  $t = 0.70$  $t = 0.80$
5. Hyperbolic Structure

Finally the relaxed model allows serious simulations (by re-vert ing time):