Emergent behaviors of the swarmalator model for position-phase aggregation

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Feb 15, 2019
1. What is “swarmalator”?
2. Kuramoto synchronization model
3. Swarm and Sync
What is “swarmalator”?

Swarmalator = Swarm + Oscillator

What is “swarmalator”?

Kuramoto synchronization model

Swarm and Sync

- Swarm
● Synchronization
Swarmalator model

\[
\dot{x}_i = \omega_i + \frac{1}{N} \left( \sum_{j=1}^{N} \frac{x_j - x_i}{|x_j - x_i|} (A + J \cos(\theta_j - \theta_i)) - B \frac{x_j - x_i}{|x_j - x_i|^2} \right)
\]

\[
\dot{\theta}_i = \nu_i + \frac{K}{N} \sum_{j=1}^{N} \frac{\sin(\theta_j - \theta_i)}{|x_j - x_i|}
\]

for \( x_i \in \mathbb{R}^2 \) and \( \theta_i \in \mathbb{T} \).

Swarm

By ignoring the phase variable, we attain

\[
\frac{dx_i}{dt} = \frac{1}{N} \sum_{\substack{j \in \mathcal{N} \\ j \neq i}} \nabla E(x_j - x_i), \quad E(x) = \frac{|x|^{2-\alpha}}{2 - \alpha} - \frac{|x|^{2-\beta}}{2 - \beta},
\]

which is the swarming of particles with attractive/repulsive interactions via power-law potential.

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Swarming model for predator-prey interaction

\[ \dot{x}_i = \omega_i + \frac{1}{N} \left( \sum_{\substack{j=1 \atop j \neq i}}^{N} a(x_j - x_i) - \frac{x_j - x_i}{|x_j - x_i|^2} \right) - b \frac{z - x_i}{|z - x_i|^2} \]

\[ \dot{z} = \frac{C}{N} \sum_{j=1}^{N} \frac{x_j - z}{|x_i - z|^p} \]

Kuramoto synchronization model

- Kuramoto model:

\[
\dot{\theta}_i = \Omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)
\]

\[\Theta = (\theta_1, \cdots, \theta_N),\]

\[\Theta(0) = \Theta_0\]

\[\Omega_i: \text{Natural frequency}\]

\[K: \text{Coupling strength}\]

- **Identical** oscillator system: \(\Omega_i = \Omega\) for all \(i = 1, \cdots, N\).

- **Non-identical** oscillator system: if there exists \(i\) and \(j\) such that \(\Omega_i \neq \Omega_j\).
Definition

Let $\Theta$ be a smooth solution to the system (2.1).

1. The dynamical solution $\Theta = \Theta(t)$ shows **asymptotically complete phase synchronization** if

$$\lim_{t \to \infty} |\theta_i - \theta_j| = 0, \quad \text{for all } i, j \in \{1, \cdots, N\}.$$

2. The dynamical solution $\Theta = \Theta(t)$ shows **phase locked state** if

$$\lim_{t \to \infty} |\theta_i - \theta_j| = \theta_{ij}^\infty, \quad \text{for all } i, j \in \{1, \cdots, N\}.$$

3. The dynamical solution $\Theta = \Theta(t)$ shows **asymptotically complete frequency synchronization** if

$$\lim_{t \to \infty} |\dot{\theta}_i - \dot{\theta}_j| = 0, \quad \text{for all } i, j \in \{1, \cdots, N\}.$$
Let $\Theta = (\theta_1, \cdots, \theta_N)$. We have an analytic potential:

$$V(\Theta) = -\sum_{i=1}^{N} \Omega_i \theta_i + \frac{K}{2N} \sum_{i,j=1}^{N} (1 - \cos(\theta_j - \theta_i))$$

Kuramoto model can be rewritten into

$$\dot{\Theta} = -\nabla V(\Theta).$$

By an application of Łojasiewicz inequality, the uniform boundedness of fluctuation implies the convergence to the phase-locked state.
Theorem (Ha, Kim, and Ryoo)

Suppose that the initial configuration $\Theta_0$ and natural frequencies $\Omega_i$ satisfy

$$r_0 > 0, \quad \theta_{i0} \neq \theta_{j0}, \quad 1 \leq i, j \leq N, \quad \max_{1 \leq i \leq N} |\Omega_i| \leq L < \infty.$$  

Then, there exists a large coupling strength $K_\infty > 0$ such that, for any solution $\Theta = (\theta_1, \cdots, \theta_N)$ to system (2.1) with initial data $\Theta_0$ and $K \geq K_\infty$, there exists a unique phase-locked state $\Theta_\infty$ such that

$$\lim_{t \to \infty} \|\Theta(t) - \Theta_\infty\|_\infty = 0.$$  

$$r_0 = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_{j0}} \right|$$
When the number of oscillator $N$ is very large, it is efficient to describe the system with the kinetic equation.

Let $f(\theta, \Omega, t)$ be the one-particle distribution function. Then, the mean-field limit of the Kuramoto model is presented by

$$\partial_t f + \partial_\theta (v[f] f) = 0, \quad (\theta, \Omega) \in \mathbb{T} \times \mathbb{R}, \ t > 0,$$

$$v[f](\theta, \Omega, t) = \Omega - KL[\rho],$$

$$L[\rho] := \int_{\mathbb{T}} \sin(\theta - \theta^*) \rho(\theta^*, t) d\theta^*$$

where

$$\rho(\theta, t) := \int_{\mathbb{R}} f(\theta, \Omega, t) d\Omega, \quad t > 0.$$

This system is called the Kuramoto-Sakaguchi equation.
J. A. Carrillo et al. show that the measure valued solution for the identical oscillators converges to the Dirac measure which implies the phase synchronization. This result assumed that the support of initial configuration is confined in a half circle.

D. Amadori, S.-Y. Ha and J. Park show the well-posedness of BV solution for the identical Kuramoto-Sakaguchi equation by using the wave-front tracking method.

Ha, Kim, Morales, and Park show the emergence of phase synchronization for identical oscillators under generic $C^1$ initial configuration. For non-identical oscillators, we show the asymptotic concentration of phase into an interval for sufficiently large coupling strength.
Swarm and Sync

- **Swarmalator model**

\[
\dot{x}_i = \omega_i + \frac{1}{N} \left( \sum_{\substack{j=1 \ j \neq i}}^{N} \frac{x_j - x_i}{|x_j - x_i|} \left( A + J \cos(\theta_j - \theta_i) \right) - B \frac{x_j - x_i}{|x_j - x_i|^2} \right) \\
\dot{\theta}_i = \nu_i + \frac{K}{N} \sum_{\substack{j=1 \ j \neq i}}^{N} \frac{\sin(\theta_j - \theta_i)}{|x_j - x_i|} 
\]

for \( x_i \in \mathbb{R}^2 \) and \( \theta_i \in \mathbb{T} \).

What is “swarmalator”?

Kuramoto synchronization model

Swarm and Sync

Static sync

Static async

Static phase wave

Splintered phase wave

Active phase wave

http://usediscretion.blogspot.com/


**Swarmalator model**

\[
\begin{align*}
\dot{x}_i &= \omega_i + \Phi(x_j - x_i, \theta_j - \theta_i) \\
\dot{\theta}_i &= \nu_i + \Psi(x_j - x_i, \theta_j - \theta_i)
\end{align*}
\]  

(3.3)

where

\[
\Phi(x_j - x_i, \theta_j - \theta_i) = \frac{1}{N} \left( \sum_{\substack{j=1 \atop j \neq i}}^{N} \frac{x_j - x_i}{|x_j - x_i|^{\alpha}} (A + J \cos(\theta_j - \theta_i)) - B \frac{x_j - x_i}{|x_j - x_i|^{\beta}} \right)
\]

\[
\Psi(x_j - x_i, \theta_j - \theta_i) = \frac{K}{N} \sum_{\substack{j=1 \atop j \neq i}}^{N} \sin(\theta_j - \theta_i) \frac{1}{|x_j - x_i|^{\gamma}}
\]

for \(x_i \in \mathbb{R}^d\) and \(\theta_i \in \mathbb{T}\). We assume

\[1 \leq \alpha < \beta\]

(3.4)
• Basic property

Note that the coupling functions $\Phi$ and $\Psi$ satisfy skew-symmetric properties under the index exchange $(i, j) \leftrightarrow (j, i)$:

$$
\Phi(x_j - x_i, \theta_j - \theta_i) = -\Phi(x_i - x_j, \theta_i - \theta_j), \\
\Psi(x_j - x_i, \theta_j - \theta_i) = -\Psi(x_i - x_j, \theta_i - \theta_j)
$$

so that the total double sums will vanish:

$$
\sum_{i,j \in \mathbb{N} \atop i \neq j} \Phi(x_j - x_i, \theta_j - \theta_i) = 0, \quad \sum_{i,j \in \mathbb{N} \atop i \neq j} \Psi(x_j - x_i, \theta_j - \theta_i) = 0. \quad (3.6)
$$
Due to the singular terms $\frac{1}{|x_j - x_i|}$ on the right hand side, if there is a collision between particles, the well-posedness is not guaranteed. We first show the collision avoidance property of the system.

**Theorem**

Suppose that the parameters $\alpha$, $\beta$ satisfy (3.4) and the initial data $(X_0, \Theta_0)$ is non-collisional, i.e.,

$$\min_{1 \leq i, j \leq N} |x_{i0} - x_{j0}| > 0.$$

Then, there exists a global solution $(X, \Theta)$ to (3.3) with

$$x_i(t) \neq x_j(t), \quad \text{for all} \quad i \neq j, \quad t \in (0, \infty).$$
(Sketch of proof) Suppose there is a first finite time collision at $t_0$. Then, we define a collision set

$$C := \left\{ i \in \mathcal{N} : \lim_{t \to t_0^-} |x_i(t) - x_\ell(t)| = 0 \right\}$$

and a functional

$$\mathcal{X}_C := \sqrt{\sum_{i,j \in C} |x_i(t) - x_j(t)|^2}.$$ 

By using this, we derive a differential inequality for $\mathcal{X}_C$:

$$\frac{d}{dt} \mathcal{X}_C \geq C \mathcal{X}_C, \quad t \in (t_0 - \varepsilon^*, t_0), \quad \varepsilon^* \ll 1. \quad (3.7)$$

This yields

$$\mathcal{X}_C(t_0) \geq \mathcal{X}_C(t_0 - \varepsilon^*) e^{C\varepsilon^*}.$$ 

Since $\mathcal{X}_C(t_0) = 0$, this yields

$$\mathcal{X}_C(t_0 - \varepsilon^*) = 0,$$

which contradicts to the first collision.
Moreover, we have a uniform in time lower bound for interparticle distances.

**Theorem**

Suppose that the parameters $\alpha$, $\beta$ satisfy (3.4) and the initial data $(X_0, \Theta_0)$ is non-collisional, i.e.,

$$\min_{1 \leq i, j \leq N} |x_{i0} - x_{j0}| > 0,$$

and let $(X, \Theta)$ be a solution to (3.3). Then, there exists a positive constant $\delta_1$ such that

$$\inf_{0 \leq t < \infty} \min_{i, j} |x_i(t) - x_j(t)| \geq \delta_1.$$
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(idea of proof) We denote the inter particle distance as \( D_{ij} := |x_i - x_j| \). Then, for each time we have an ordering of distances.

\[
D_1(t) := D_{i_1j_1}(t) \leq D_2(t) := D_{i_2j_2}(t) \leq \cdots \leq D_Q(t) := D_{i_Qj_Q}(t),
\]

(3.8)

where \( Q \) is the total number of pairs \( Q = \frac{N(N-1)}{2} \). This ordering maintains for some time intervals and we have

\[
[0, \infty) = \bigcup_{l=1}^{\infty} T_l, \quad T_1 = [t_0, t_1) = [0, t_1), \quad T_l := [t_{l-1}, t_l), \quad l \geq 2,
\]

For each interval, we have a lower bound for a maximal distance

\[
\inf_{t \in T_1^o} D_Q(t) \geq \delta_Q := \min \left\{ D_Q(0), \left( \frac{m_r}{2M_a} \right)^{\frac{1}{\beta-\alpha}}, \left( \frac{m_r}{ND(W)} \right)^{\frac{1}{\beta-1}} \right\}.
\]

By the induction forward in time, we have a uniform lower bound.

\[
\inf_{0 \leq t < \infty} D_Q(t) \geq \delta_Q.
\]

We use the lower bound \( \delta_Q \) for \( D_Q(t) \) to attain a lower bound \( \delta_{Q-1} \) for \( D_{Q-1}(t) \). Thus, by descending induction, we have a uniform lower bound \( \delta_1 \) for \( D_1(t) \).
We can prove the swarming under some conditions.

**Theorem**

Suppose that the parameters, natural velocities \( \{w_i\} \) and the initial data \((X_0, \Theta_0)\) satisfy one of the frameworks \((F1) - (F2)\) and let \((X, \Theta)\) be a solution to (3.3). Then, there exists a positive constant \(\delta_\infty\) such that

\[
\sup_{0 \leq t < \infty} D(t) \leq \delta_\infty.
\]
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\[
\gamma_1 := \frac{2m_a}{N}, \quad \gamma_2 := \frac{2M_r}{N}, \quad \gamma_3 := \mathcal{D}(W).
\]

- (F1): For \( \alpha = 1 \),

\[
\mathcal{D}(0) < \infty, \quad \delta_1 > \left( \frac{M_r}{m_a} \right)^{\beta - \alpha}, \quad \gamma_1 > \gamma_3.
\]

- (F2): For \( \alpha > 1 \),

\[
\mathcal{D}(0) < y^*, \quad \delta_1 > \left( \frac{M_r}{m_a} \right)^{\beta - \alpha}, \quad \text{and}
\]

\[
- \gamma_1 \left( \frac{\gamma_1(\alpha - 1)}{\gamma_2(\beta - 1)} \right)^{(\beta - \alpha)(\alpha - 1)} + \gamma_2 \left( \frac{\gamma_1(\alpha - 1)}{\gamma_2(\beta - 1)} \right)^{(\beta - \alpha)(\beta - 1)} + \gamma_3 \leq 0,
\]

where \( y^* \) is the largest root of the equation:

\[
-\gamma_1 y^{1-\alpha} + \gamma_2 y^{1-\beta} + \gamma_3 = 0.
\]
• While the position of the particles are bounded, we can attain a practical synchronization result for the phase dynamics.

**Theorem**

Suppose that the parameters \( \alpha, \beta \), natural velocities \( \{w_i\} \) and the initial data \((X_0, \Theta_0)\) satisfy one of the frameworks \((\mathcal{F}1) - (\mathcal{F}2)\) and let \((X, \Theta)\) be a solution to (3.3) with the initial data \((X_0, \Theta_0)\). Moreover, we assume the following additional assumptions:

\[
0 < D(\Theta_0) < \pi, \quad \kappa > \frac{D(\Omega)\delta_y}{\sin D(\Theta_0)}.
\]

Then, we have a practical synchronization estimate:

\[
D(\Theta(t)) \leq D(\Theta_0)e^{-\frac{\kappa R_0}{\delta_y} t} + \frac{D(\Omega)\delta_y}{\kappa R_0} \left(1 - e^{-\frac{\kappa R_0}{\delta_y} t}\right), \quad t \geq 0,
\]

where \( R_0 := \frac{\sin D(\Theta_0)}{D(\Theta_0)} \).

We note that the phase shows the phase synchronization for identical oscillators.
For identical oscillators, we define the analytic potential:

\[
V_\alpha := \begin{cases} 
\sum_{i=1}^{N} w_i \cdot x_i + \frac{1}{N} \sum_{i \neq j} \log(|x_i - x_j|)(1 + J \cos(\theta_j - \theta_i)), & \alpha = 2, \\
\sum_{i=1}^{N} w_i \cdot x_i + \frac{1}{N} \sum_{i \neq j} \frac{|x_i - x_j|^{2-\alpha}}{2 - \alpha}(1 + J \cos(\theta_j - \theta_i)), & \alpha \neq 2,
\end{cases}
\]

\[
V_\beta := \begin{cases} 
- \frac{1}{N} \sum_{i \neq j} \log(|x_i - x_j|), & \beta = 2, \\
\frac{1}{N} \sum_{i \neq j} \frac{|x_i - x_j|^{2-\beta}}{\beta - 2}, & \beta \neq 2.
\end{cases}
\]

We immediately obtain the gradient flow structure

\[
\dot{X} = -\nabla_X V(X, \Theta), \quad V = V_\alpha + V_\beta.
\]
Although the potential depends on both position and phase, the phase dynamics shows the synchronization for identical oscillators. Thus, we can use the gradient flow structure to show the convergence of the position by using the boundedness of the solution.

**Theorem**

Suppose that the parameters $\alpha$, $\beta$, natural velocities $\{w_i\}$ and the initial data $(X_0, \Theta_0)$ satisfy one of the frameworks $(F1) - (F2)$ and let $(X, \Theta)$ be a solution to (3.3) with the initial data $(X_0, \Theta_0)$ such that $0 < D(\Theta_0) < \pi$, $\kappa > 0$ and $D(\Omega) = 0$. Then there exists $X^\infty$ such that

$$\lim_{t \to \infty} X(t) = X^\infty.$$  (3.11)
Thank You